

Background-Field Formalism in Nonperturbative QCD

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Abstract

The background-field formalism is used extensively in fundamental approaches to QCD to explore hadronic matrix elements of various currents. While the lattice QCD approach is formulated in the fully-interacting Hilbert space, which includes both QCD and background field interactions, the QCD sum-rule formalism is traditionally developed in the pure-QCD Hilbert space. The latter approach encounters difficulties with excited state contaminations which are not exponentially suppressed and requires extrapolations to isolate the desired physics. Proponents of the pure-QCD Hilbert space formalism used in the QCD sum-rule approach have criticized the lattice QCD approach as neglecting important physics. In this letter, the equivalence of the two approaches is established and the flaws in the former criticisms are resolved. Finally, the application of the fully-interacting Hilbert space formalism to the study of electromagnetic polarizabilities is outlined.

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The background-field formalism of field theoretic approaches to hadron phenomenology has been used extensively to resolve the static properties of hadrons. For example, early lattice QCD calculations of baryon magnetic moments [1,2] introduced a background magnetic field into the gauge field configurations. The magnetic moments were extracted by calculating the energy shift of the ground state in the presence of the background field which is proportional to $\boldsymbol{\mu} \cdot \mathbf{B}$. The QCD sum-rule approach has also exploited the background-field formalism to study hadron properties such as magnetic moments [3,4]. However, the description for propagating hadrons in the presence of the background field is handled differently in these two approaches.

In the lattice QCD approach, the effects of the background field are treated in a non-perturbative sense. The propagating quarks experience the background field at each link throughout the lattice volume such that complex motion such as the zero-point orbital motion can evolve. When inserting a complete set of hadron states to describe the propagator, the Hilbert space of the full Hamiltonian, including both QCD and background-field effects, arises naturally.

On the other hand, only the linear response to the background field is considered in the traditional QCD sum-rule approach. In this case the effects of the background field are treated perturbatively at the phenomenological level. It is worth emphasizing that the background-field QCD sum-rule approach, usually adopted in the literature, is identical to the QCD sum-rule approach based on direct three-point function considerations, and the introduction of the background field is only for bookkeeping. In the QCD sum-rule background-field formalism, one can select the Hilbert space of either the full Hamiltonian or the normal QCD Hamiltonian. However, as we will show, one ultimately needs to resort to the Hilbert space of the normal QCD Hamiltonian in order to extract the hadron properties of interest.

A well-known problem that arises in the QCD sum-rule background-field formalism is the unwanted physics associated with transitions from the ground state hadron to excited states. The contributions of these transitions are not exponentially suppressed (after Borel transformation) relative to the ground state contribution which contains the ground state property of interest. This drawback is general to any QCD sum-rule calculation based on three-point functions.

In principle, there are infinitely many transition terms as there are infinitely many excited states, and they should be included in the QCD sum-rule calculations. Fortunately, the polynomial (in Borel mass) behavior of the ground state signal is different from those of the transition terms. The usual approximation is to introduce a new unknown phenomenological parameter (independent of Borel mass) accounting for the sum over the contributions from all the transitions between the ground state and the excited states. This new parameter is extracted from the sum rules, along with the ground state property. The hope is that a sufficiently large valid Borel regime exists such that the interplay of the two parameters may be resolved. While this approximation has been used in earlier QCD sum-rule studies, it has been noticed recently that the parameter representing the transitions is in general dependent on the Borel mass, which may have a sizable impact on the extracted ground state properties [5,6].

In Ref. [3] Ioffe and Smilga argued that the transition terms are also problematic in lattice QCD calculations employing the background-field approach. They claim that the

transition contributions were overlooked in Ref. [1,2] and may be as large as 1/3 of the signal associated with the ground state magnetic moment.

Once the sequential source technique [7] was established for calculating the propagators encountered in three-point functions, the consideration of hadronic matrix elements became the method of choice for determining hadron properties such as electromagnetic form factors [8–11]. However, given the difficulties associated with calculating four-point functions in lattice field theory [12], the background field formalism may be the best way to make contact with nucleon electromagnetic polarizabilities and other aspects of the Compton scattering program of TJNAF.

In this letter we will examine the relationship between the two apparently different approaches of lattice QCD and QCD sum rules. We will illustrate how the problematic contaminations of the transition terms encountered in the QCD sum-rule formalism are specific to the QCD sum-rule approach. There, only the linear response term in the background field is considered and hence the Hilbert space of the normal QCD Hamiltonian is employed. Moreover, we will illustrate that to first order in the background field strength the two approaches are indeed equivalent, and that the problematic excited state contaminations simply reflect the unfortunate but necessary resort to a Hilbert space that is not the fully-interacting Hilbert space.

In anticipation of applying the background-field formalism to the study of polarizabilities in lattice field theory, we will formulate our discussion in Euclidean space-time. The results are easily transformed to the QCD sum-rule formalism.

Consider the change in the two-point function due to the presence of a background field. If δH is the change in the Hamiltonian reflecting the background field effects, then to leading order in δH [9] we have

$$\delta G(t_2; \mathbf{p}, \mathbf{q}) = \int_0^{t_2} dt_1 \int d^3 x_2 d^3 x_1 e^{-i\mathbf{p}\cdot\mathbf{x}_2} e^{+i\mathbf{q}\cdot\mathbf{x}_1} \langle \Omega | T [\chi(x_2) \delta H(x_1) \bar{\chi}(0)] | \Omega \rangle. \quad (1)$$

Here \mathbf{q} denotes the three-momentum of the interaction and \mathbf{p} denotes the momentum of the final state. χ is a hadron interpolating field constructed from local quark and gluon operators identifying the spin and isospin of the hadron under investigation. Since intermediate states propagate on shell, a variety of momentum transfers may be studied. However, our focus here is on zero momentum transfer which is easily accessed via $\mathbf{p} = \mathbf{q} = 0$. Hence we focus on

$$\delta G(T) = \int_0^T dt \langle \Omega | T [\chi(T) \delta H(t) \bar{\chi}(0)] | \Omega \rangle, \quad (2)$$

where $T = t_2$, $t = t_1$ and the integrals over spatial coordinates are implicit.

To make contact with the QCD sum-rule formalism, we proceed by inserting a complete set of states in the normal QCD Hilbert space denoted by $|N\rangle$ and $|N^*\rangle$ indicating ground and excited states, respectively.

$$\begin{aligned} \delta G(T) = & \int_0^T dt \left[\langle \Omega | \chi | N \rangle e^{-M_N(T-t)} \langle N | \delta H | N \rangle e^{-M_N t} \langle N | \bar{\chi} | \Omega \rangle \right. \\ & \left. + \sum_{N^* \neq N} \langle \Omega | \chi | N^* \rangle e^{-M_{N^*}(T-t)} \langle N^* | \delta H | N \rangle e^{-M_{N^*} t} \langle N | \bar{\chi} | \Omega \rangle \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{N^* \neq N} \langle \Omega | \chi | N \rangle e^{-M_N(T-t)} \langle N | \delta H | N^* \rangle e^{-M_{N^*}t} \langle N^* | \bar{\chi} | \Omega \rangle \\
& + \text{exponentially suppressed terms.}
\end{aligned} \tag{3}$$

Since t is integrated from 0 to T the interaction Hamiltonian $\delta H(t)$ can get arbitrarily close to the source ($\bar{\chi}$) or sink (χ). Therefore, even as $T \rightarrow \infty$, terms that include contributions from hadron resonances are *not* exponentially suppressed compared to the ground state property one is interested in. In fact, explicit integration yields

$$\begin{aligned}
\delta G(T) &= \langle \Omega | \chi | N \rangle \langle N | \delta H | N \rangle \langle N | \bar{\chi} | \Omega \rangle T e^{-M_N T} \\
&+ \sum_{N^* \neq N} \left\{ \left[\langle \Omega | \chi | N^* \rangle \langle N^* | \delta H | N \rangle \langle N | \bar{\chi} | \Omega \rangle \right. \right. \\
&\quad \left. \left. + \langle \Omega | \chi | N \rangle \langle N | \delta H | N^* \rangle \langle N^* | \bar{\chi} | \Omega \rangle \right] \frac{e^{-(M_{N^*} - M_N)T} - 1}{M_N - M_{N^*}} \right\} e^{-M_N T} \\
&+ \text{exponentially suppressed terms.}
\end{aligned} \tag{4}$$

The latter term containing the transition matrix elements essentially falls off as $\exp(-M_N T)$, demonstrating that the transition contribution is only power law suppressed in T . This excited-state contamination was first encountered in the QCD sum-rule approach. The usual approximation is to introduce a new fit parameter accounting for the quantity in braces and use the different T (or squared Borel mass, M^2 , in QCD sum rules) dependence to isolate the ground state signal. It is not clear how this treatment affects the predictive ability of the QCD sum-rule approach, as a realistic uncertainty analysis [13] has not been done. Nevertheless, error estimates of the standard 10% are claimed.

Eq. (4) also shows how a naive evaluation of background-field two-point functions in Euclidean lattice QCD may contain large contaminations from transitions to the N^* as well. While lattices are large enough that exponentially suppressed terms are negligible, this cannot be said about power law suppressed pieces. We note however for clarity, that these problems are not encountered in the standard treatment of three-point functions on the lattice. There, the intermediate point t is fixed at large values to ensure exponential suppression of excited states prior to interaction with the probing current. The final time T is taken to be $\gg t$ to once again ensure isolation of the ground state signal following interaction with the current.

Now consider the two-point function in the presence of the background field to all orders in δH

$$G(T) = \langle \Omega | T [\chi(T) \bar{\chi}(0)] | \Omega \rangle. \tag{5}$$

This is the approach adopted in background-field lattice QCD calculations. If one denotes by $|N'\rangle$, $|N^{*'}\rangle$ the exact eigenstates in the fully-interacting Hilbert space, then

$$G(T) = \langle \Omega | \chi | N' \rangle \langle N' | \bar{\chi} | \Omega \rangle e^{-M_{N'} T} + \text{exponentially suppressed terms.} \tag{6}$$

In Eq.(6) the transitions to excited states are hidden in the implicit dependence of the basis states on the background field, as reflected in the first order perturbation theory formula

$$|N'\rangle = |N\rangle + \sum_{N^* \neq N} \frac{|N^*\rangle \langle N^*| \delta H |N\rangle}{M_N - M_{N^*}} + \mathcal{O}(\delta H^2), \quad (7)$$

yielding the dependence of the “coupling constants” on the background field

$$\langle \Omega | \chi | N' \rangle = \langle \Omega | \chi | N \rangle + \sum_{N^* \neq N} \frac{\langle \Omega | \chi | N^* \rangle \langle N^* | \delta H | N \rangle}{M_N - M_{N^*}} + \mathcal{O}(\delta H^2). \quad (8)$$

The equivalence of Eq.(4) and Eq.(6) can then be verified by utilizing the perturbative expansion of the mass

$$M_{N'} = M_N + \langle N | \delta H | N \rangle + \mathcal{O}(\delta H^2), \quad (9)$$

and expanding the exponential to leading order

$$e^{-\langle N | \delta H | N \rangle T} = 1 - \langle N | \delta H | N \rangle T + \mathcal{O}(\delta H^2). \quad (10)$$

We observe that the transition contaminations originate from the response of the hadron wave function to the background field as shown in Eq. (7). This elementary observation clearly indicates how one may avoid the problem of contamination from excited states. First one calculates the two point function of Eq. (5) in the presence of the background field. Rather than introducing pure-QCD parameters such as the nucleon mass or nucleon coupling constants of the interpolators, one simply extracts the mass of the ground state in the fully-interacting Hilbert space by fitting the slope of the logarithm of the lattice two-point correlation function. By determining the dependence of the ground state mass as a function of background-field strength, the magnetic moment may be extracted from the slope as $|\mathbf{B}| \rightarrow 0$. This has always been the procedure used by lattice QCD practitioners in the background-field formalism.

On the other hand, if one evaluates the two-point function only to leading order in δH , as in the usual QCD sum-rule formalism, the transition contamination will be unavoidable. This can be seen by expanding the exponential of (6) to leading order

$$\delta G(T) = \langle \Omega | \chi | N' \rangle \langle N' | \bar{\chi} | \Omega \rangle \langle N | \delta H | N \rangle T e^{-M_N T}. \quad (11)$$

It is impossible to resolve both the unknown couplings, $\langle \Omega | \chi | N' \rangle \langle N' | \bar{\chi} | \Omega \rangle$, and the matrix element of interest, $\langle N | \delta H | N \rangle$. Hence, one is forced to expand the fully-interacting Hilbert space couplings in order to make contact with the normal QCD Hilbert space couplings determined from other correlation functions. In doing so, one must confront the contaminations of excited state contributions which are not exponentially suppressed.

Finally, it is tempting to adopt Eq. (6) in the QCD sum-rule formalism. This, however, requires a new method to evaluate the two-point function to all orders in δH . Such a method is not available to date. Nevertheless, it will be interesting to determine the uncertainty associated with $\langle N | \delta H | N \rangle$ in the QCD sum-rule approach. It is well known that the interpolating field couplings show much larger relative errors than the exponential decay parameter such as the mass [13]. The matrix element $\langle N | \delta H | N \rangle$ and the couplings play

very similar roles as illustrated in (4). Hence it is important to establish the true predictive ability of QCD sum rules for hadron matrix elements. Research in this direction is in progress [14,15].

Having established the viability of the background field formalism, it is interesting to consider the possibility of determining second order effects of the background field. Electromagnetic polarizabilities may be extracted from (6) by simply expanding the extracted mass to second order in δH . For example, the energy of a neutron in a constant background magnetic field is given by

$$E = M_N - \mu_n B + 4\pi \frac{\beta_n}{2} B^2 + \mathcal{O}(B^3). \quad (12)$$

To extract both magnetic moments and magnetic polarizabilities, one should select a field strength such that

$$\mu_n B \sim 4\pi \frac{\beta_n}{2} B^2. \quad (13)$$

By considering both spin polarizations, the polarizability and the magnetic moment may be isolated individually. In order to use quark propagators in the external field for both u and d quarks, the magnetic field strengths should be selected in the ratio $-1 : +2 : -4 : +8$ providing three different field strengths for study. To satisfy the criteria of (13) at the intermediate field strength, one requires a minimum field strength of

$$e B = \frac{e}{4\pi} \frac{\mu_n}{\beta_n} \simeq 5 \text{ fm}^2, \quad (14)$$

where $e^2/4\pi = 1/137$ and experimental values of $\mu_n = -1.913 \mu_N$ and $\beta_n = 0.3 (10^{-3}) \text{ fm}^3$ [16] have been used for the estimate. The relationship for the smallest uniform magnetic field available on a cubic lattice of cross-sectional area A is [17]

$$e B = \frac{2\pi}{A q}, \quad (15)$$

where $q = 2/3$ or $-1/3$ for u and d quarks respectively. To accommodate d quarks in the minimum uniform field of (14) requires an area $A \sim 4 \text{ fm}^2$. We note that with improved lattice QCD actions at lattice spacings of $\sim 0.25 \text{ fm}$ one requires a small 8^3 lattice to satisfy this criteria.

However the magnetic moment interaction energy associated with (14) is 200 MeV for the minimum field strength and grows to 800 MeV. Similarly the polarizability interaction energies range from 100 MeV to 1600 MeV. These values are too large to make contact with the experimentally measured quantities. Hence, one needs to consider somewhat smaller field strengths. Since the interaction energy for polarizabilities varies as the square of the field strength, only slightly smaller field strengths are required. For example, at lattice spacings of 0.23 fm, polarizability interaction energies of 25, 100, and 400 MeV are provided by a 12^3 lattice. With modest supercomputer resources, these small lattices will provide the opportunity to move beyond the quenched approximation and investigate full QCD in which dynamical fermions are included. Since current lattice simulations use quark masses which

are above the physical quark masses, the remaining key source of systematic uncertainty will lie in extrapolations to the physical quark masses.

Hence, a determination of magnetic polarizabilities is readily possible today. It will be interesting to examine the effectiveness of $\mathcal{O}(a^2)$ -improved lattice QCD actions for describing magnetic moments and polarizabilities. Evaluations based on the hadron spectrum are encouraging [18–20].

In summary, we have illustrated how Ioffe and Smilga’s criticisms [3] of the lattice QCD calculations of Refs. [1,2] are ill-founded. In fact, the equivalence of the two approaches has been established. It will be interesting to reevaluate the utility of the background field approach in light of the many advances made in lattice field theory since 1982. Our hope is that clarification of these issues will generate a renewed interest in the background field formalism as a viable method for the extraction of hadron properties from QCD.

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